5th Benelux Mathematical Olympiad

Dordrecht, 26–28 April 2013

Problem 1. Let $n \geqslant 3$ be an integer. A frog is to jump along the real axis, starting at the point 0 and making n jumps: one of length 1, one of length 2, ..., one of length n. It may perform these n jumps in any order. If at some point the frog is sitting on a number $a \leqslant 0$, its next jump must be to the right (towards the positive numbers). If at some point the frog is sitting on a number $a > 0$, its next jump must be to the left (towards the negative numbers). Find the largest positive integer k for which the frog can perform its jumps in such an order that it never lands on any of the numbers 1, 2, ..., k.

Problem 2. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) + y \leqslant f(f(f(x)))$$

holds for all $x, y \in \mathbb{R}$.

Problem 3. Let $\triangle ABC$ be a triangle with circumcircle Γ, and let I be the center of the incircle of $\triangle ABC$. The lines AI, BI and CI intersect Γ in $D \neq A$, $E \neq B$ and $F \neq C$. The tangent lines to Γ in F, D and E intersect the lines AI, BI and CI in R, S and T, respectively. Prove that

$$|AR| \cdot |BS| \cdot |CT| = |ID| \cdot |IE| \cdot |IF|.$$

Problem 4.

a) Find all positive integers g with the following property: for each odd prime number p there exists a positive integer n such that p divides the two integers

$$g^n - n \quad \text{and} \quad g^{n+1} - (n+1).$$

b) Find all positive integers g with the following property: for each odd prime number p there exists a positive integer n such that p divides the two integers

$$g^n - n^2 \quad \text{and} \quad g^{n+1} - (n+1)^2.$$

Language: English

Time available: 4.5 hours
Each problem is worth 7 points